Yang-Mills Instantons with Lorentz Violation

Don Colladay and Pat McDonald

New College of Florida

Sarasota, FL, 34243, U.S.A.

An analysis is performed of instanton configurations in pure Euclidean Yang-Mills theory containing small Lorentz-violating perturbations that maintain gauge invariance. Conventional topological arguments are used to show that the general classification of instanton solutions involving the topological charge is the same as in the standard case. Explicit solutions are constructed for general gauge invariant corrections to the action that are quadratic in the curvature. The value of the action is found to be unperturbed to lowest order in the Lorentz-violating parameters.

I. INTRODUCTION

As is well known, solving pure Yang-Mills theory involves a complicated set of nonlinear partial differential equations. Using a series of clever arguments, some exact solutions to the pure Yang-Mills field equations formulated in four-dimensional Euclidean space were first constructed in the mid seventies [1]. The complete set of finite action solutions was eventually classified using what is now known as the ADHM construction [2]. Subsequently, instanton physics has stimulated much research in both physics and mathematics [3].

In pure four-dimensional Yang-Mills theory, Lorentz symmetry and renormalizability coupled with gauge invariance implies that the Lagrange density naturally takes the form of the trace of the square of the curvature tensor. If pure Yang-Mills theory arises as the low energy limit of some more fundamental theory, it is possible that real physical fields obey a slightly modified version of the conventional equations in which some of the underlying symmetries are spontaneously broken. Specifically, Lorentz and CPT invariance, as well as gauge invariance can be affected [4].

The original motivation for this possibility arose in string theory [5], and more recently has been analyzed within the context of noncommutative geometry [6]. A framework called the Standard Model Extension (SME) incorporates general fundamental symmetry violations that are consistent with coordinate reparametrization invariance[1] within the context of quantum field theory [7]. Usually it is convenient to restrict the full range of possible violations to maintain certain subgroups of the original symmetry group. For instance, translational invariance, gauge invariance and power-counting renormalizability are typically assumed to avoid many of the potential inconsistencies that may arise without these assumptions. This restriction produces a minimal version of the full SME.

The aim of this paper is to analyze the instanton solutions for a Yang-Mills action in the presence of Lorentz violation. The main result is that the general classification of the instanton solutions involving the topological charge still applies when Lorentz

---

[1]Geometrically, the symmetry violations can be defined using fixed sections of appropriate fibre bundles.

violation is present. In addition, the value of the Euclidean action is found to be invariant to lowest order in the Lorentz-violating perturbations. Specific calculations are performed within the framework of the minimal SME, but some of the results are in fact more general. In section II, the notation and conventions are described. The existence of static solutions in arbitrary dimensions is examined in section III. Section IV contains the general theory of instantons with Lorentz violation, while section V restricts to the specific example of SU(2) instantons with unit winding number. Section VI summarizes the results. The appendix contains an exact solution for the perturbed instantons in the presence of a spatially isotropic Lorentz-violating background tensor.

II. NOTATION AND CONVENTIONS

The conventions used for the Yang-Mills gauge theory are presented in this section. Let G be a compact Lie group with Lie algebra L(G). The base manifold is taken to be $M = \mathbf{R}^4$ and the gauge potential components for the principle G-bundle $P \to M$ are denoted

$$A_\mu(x) \equiv A_\mu^a(x) L_a \quad , \tag{1}$$

where the L_a are hermitian generators of a Lie algebra defined by

$$[L_a, L_b] = i C_{abc} L_c \quad , \tag{2}$$

with structure constants C_{abc} antisymmetric in all indices. The normalization of the generators is fixed by imposing

$$Tr(L_a L_b) = \tfrac{1}{2}\delta_{ab} \quad . \tag{3}$$

The associated unitary Lie group elements that generate gauge transformations are denoted by

$$U(x) = e^{-i\omega^a(x)L_a} \quad . \tag{4}$$

These act on the gauge fields via the transformation rule

$$A^\mu(x) \to U(x)A^\mu(x)U^{-1}(x) - \frac{i}{g}U(x)\partial^\mu U^{-1}(x) \tag{5}$$

The field strength tensor is defined as

$$F^{\mu\nu} = -\frac{i}{g}[D^\mu, D^\nu] \quad , \tag{6}$$

where the covariant derivative is taken as $D^\mu = \partial^\mu + igA^\mu$. The field strength transforms under gauge transformations as

$$F^{\mu\nu} \to U(x)F^{\mu\nu}U^{-1}(x) \quad . \tag{7}$$

The dual of F is defined as

$$\tilde{F}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \quad , \tag{8}$$

where the Levi-Civita tensor is defined such that $\epsilon^{0123} = +1$.

In four dimensional Minkowski space, with metric $g = diag(1, -1, -1, -1)$, The most general gauge invariant[2] and power counting renormalizable action is [7]

$$S_M(A) = -\frac{1}{2}\int d^4x Tr\left[F^{\mu\nu}F_{\mu\nu} + (k_F)_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\right.$$
$$\left. + (k_{AF})_\kappa\epsilon_{\kappa\lambda\mu\nu}(A^\lambda F^{\mu\nu} - \tfrac{2}{3}igA^\lambda A^\mu A^\nu)\right] \quad , \tag{9}$$

where the k_F and k_{AF} terms are small, constant background parameters. Gauge invariance fixes these parameters to be singlets under the action of the gauge group. The k_{AF} terms present theoretical difficulties associated with negative contributions to the energy [8] even in the Abelian case, and are therefore not considered further in this work. On the other hand, the k_F terms do not cause similar problems provided a concordant frame [9], in which the parameters are small enough, is used. The parameters k_F satisfy the symmetries of the Riemann curvature tensor[3] with vanishing total trace. This means that there are 19 independent coefficients that parameterize the violation.

III. STATIC SOLUTIONS

In the conventional case, finite-action static solutions are ruled out in all but four spatial dimensions by considering various integrals of the field strength products

---

[2]The gauge invariance of the k_{AF} term can be easily established for infinitesimal gauge transformations. Large gauge transformations may contribute nontrivially to the action.

[3]The totally antisymmetric component would contribute a term to the action that is proportional to the topological charge. This corresponds to the Lorentz-covariant θ term used in QCD.

motivated by the form of the energy momentum tensor[10]. This result also holds in the Lorentz-violating case due to an analogous argument that will now be presented.

The partially symmetrized energy momentum tensor arising from the action in Eq. (9) generalized to n spatial dimensions is given by the expression

$$\Theta^{\mu\nu} = 2Tr[-F^{\nu}{}_{\gamma}(F^{\mu\gamma} + k_F^{\mu\gamma\alpha\beta}F_{\alpha\beta}) + \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}(F^{\alpha\beta} + k_F^{\alpha\beta\lambda\kappa}F_{\lambda\kappa})] \quad , \qquad (10)$$

and explicitly satisfies $\partial_\mu\Theta^{\mu\nu} = 0$. Choosing the static gauge in which A^k is independent of time, the following constraints on finite energy solutions can be derived using the field equations

$$\int dx^n Tr F_{0k}(F^{0k} + k_F^{0k\alpha\beta}F_{\alpha\beta}) = 0 \quad , \qquad (11)$$

and

$$(n-4)\int dx^n Tr F_{ij}(F^{ij} + k_F^{ij\alpha\beta}F_{\alpha\beta}) = 0 \quad . \qquad (12)$$

Methods analogous to the ones presented in [10] have been applied to obtain the above results. These relations imply that no static solutions with nonvanishing action[4] exist when $n \neq 4$. This result is the same as the conventional situation.

IV. INSTANTON SOLUTIONS

To study the instanton solutions, the action is analytically continued to Euclidean space using imaginary time, and a new Euclidean action $S_E \equiv -iS_M$ is defined. The conventions used in this paper are obtained using the replacements $x^0 \to -ix_E^0$, $x^k \to x_E^k$, while the gauge field components are altered to $A^0 \to iA_E^0$, and $A^k \to A_E^k$. Each time component of k_F also gets multiplied by a factor of i to define its Euclidean counterpart. The Euclidean action becomes (dropping all E subscripts)

$$S(A) = +\frac{1}{2}\int d^4x Tr[(F^{\mu\nu}F^{\mu\nu}) + (k_F)^{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}] \quad , \qquad (13)$$

with metric $\delta^{\mu\nu}$. The Euler-Lagrange equations of motion (for this Euclidean action) are

$$[D^\mu, F^{\mu\nu} + k_F^{\mu\nu\alpha\beta}F^{\alpha\beta}] = 0 \quad , \qquad (14)$$

---

[4]Nontrivial static solutions with vanishing action are not expected provided k_F is small. This is because $F = 0$ is an extremum of the action, meaning that any field of order k_F must be away from the extremum.

while the Bianchi Identity

$$[D^\mu, \tilde{F}^{\mu\nu}] = 0 \quad , \tag{15}$$

follows from the definition of F in terms of the gauge potential.

The topological charge q is defined as in the usual case

$$q = \frac{g^2}{16\pi^2} \int d^4x Tr \tilde{F}^{\mu\nu} F^{\mu\nu} \quad , \tag{16}$$

and conventional arguments indicate that q remains an integer, even in the presence of Lorentz violation. Specifically, the identity

$$\tfrac{1}{4} Tr \tilde{F}^{\mu\nu} F^{\mu\nu} = \partial^\mu X^\mu \quad , \tag{17}$$

where

$$X^\mu \equiv \tfrac{1}{4} \epsilon^{\mu\nu\lambda\kappa} Tr(A^\nu F^{\lambda\kappa} - \tfrac{2}{3} ig A^\nu A^\lambda A^\kappa) \quad , \tag{18}$$

ensures that the topological charge depends only on the pure-gauge boundary conditions satisfied by the potential far away from the nonvanishing curvature of the instantons. The quantity q is therefore the first Pontryagin number that corresponds to the winding number of the map from the gauge group to the three-sphere at large $|x|$. The specific form of the action does not matter, provided that it is in fact gauge invariant, and that finiteness of the action restricts the curvature from contributing to the topological charge at the boundary. This means that the properties of the topological charge should be preserved, even in the more general case of the SME that includes nonrenormalizable, but gauge invariant corrections to the pure Yang-Mills sector. In particular, since any physical theory of noncommutative gauge fields is argued to be equivalent to a standard gauge theory in the context of the SME [6], the topological charge should remain integral in realistic noncommutative Yang-Mills theories.

For calculational purposes, it is convenient to introduce the quantity[5]

$$F'^{\mu\nu} = F^{\mu\nu} + \tfrac{1}{2} k_F^{\mu\nu\alpha\beta} F^{\alpha\beta} \quad . \tag{19}$$

The action then takes the conventional form in terms of F' to lowest order in k_F.

---

[5]Note that F' is not in general the curvature of a connection, so the theory is not automatically isomorphic to the conventional one with $k_F = 0$.

Consider the inequality

$$\tfrac{1}{2}\int d^4x Tr(F' \mp \tilde{F}')^2 \geq 0 \quad . \tag{20}$$

This implies that

$$S \geq \pm\tfrac{1}{2}\int d^4x Tr[F^{\mu\nu}\tilde{F}^{\mu\nu} + \tfrac{1}{2}(k_F^{\mu\nu\alpha\beta} + \tilde{k}_F^{\mu\nu\alpha\beta})F^{\mu\nu}\tilde{F}^{\alpha\beta}] \quad , \tag{21}$$

where $\tilde{k}_F^{\mu\nu\alpha\beta} \equiv \tfrac{1}{4}\epsilon^{\mu\nu\lambda\kappa}k_F^{\lambda\kappa\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}$ is defined as the dual to k_F. The upper sign is chosen for $q > 0$ and the lower sign for $q < 0$.

The first term is proportional to the topological charge while the second term generates a correction to the lower bound on S. Provided k_F is small, the correction term will be much smaller than the topological charge term and the perturbed instantons will be close to the conventional ones. This implies that the general classification of the instanton solutions in terms of the winding number will remain unaltered.

It is evident from the form of the correction to the lower bound that splitting the coefficients k_F according to their duality properties will be useful. This decomposition is analogous to the separation of the Riemann tensor of general relativity into a Ricci tensor and a trace-free Weyl conformal tensor. The anti-self-dual k_F components correspond to the Ricci tensor components while the self-dual k_F terms correspond to the Weyl conformal tensor. For the case $k_F = -\tilde{k}_F$, the lower bound on the action is independent of continuous perturbations of F that do not change the topological charge by an integer, and the minimum is attained for the modified duality condition

$$F' \simeq \pm\tilde{F}' \quad , \tag{22}$$

where the symbol $\simeq$ is used to denote an equality to lowest order in k_F. To construct the perturbed solutions, the potential can be expanded about the conventional ($k_F = 0$) self-dual and anti-self-dual potentials, denoted by A_{SD} and A_{ASD}. The corresponding field tensors are written similarly as F_{SD}, and F_{ASD}. It remains to show that solutions to the modified duality condition that are consistent with the Bianchi identity exist. The anti-self duality condition on k_F implies that it must be of the form

$$k_F^{\mu\nu\alpha\beta} = \Lambda_k^{[\mu[\alpha}\delta^{\nu]\beta]} \quad , \tag{23}$$

where $\Lambda_k^{\mu\nu} = \frac{1}{2}k_F^{\alpha\mu\alpha\nu}$ is a traceless, symmetric matrix that depends on the trace-components of k_F. In fact, the explicit solution can be guessed since the form of the correction to the action is related to the conventional action as described in the skewed coordinate system $\tilde{x}^\mu \equiv x^\mu + \Lambda_k^{\mu\nu}x^\nu$. These terms are exactly the ones that may be transferred to other sectors using an appropriate field redefinition [11], so it is not surprising that they yield a conventional version of pure Yang-Mills theory when described in skewed coordinates. Note that this does not imply the absence of physical effects arising from an anti-self-dual k_F term in the action. Redefining coordinates effects all fields, not just the Yang-Mills gauge potential, so if the instantons are expressed in terms of new coordinates, the Lorentz-violation will show up in the Lagrangian for other particle species that are coupled to the instantons.

The explicit form for the perturbed self-dual instanton gauge potentials are given by

$$A_+^\mu(x) \simeq A_{SD}^\mu(\tilde{x}) + \Lambda_k^{\mu\nu}A_{SD}^\nu(x) \quad , \tag{24}$$

yielding a perturbed field tensor

$$F_+^{\mu\nu} \simeq F_{SD}^{\mu\nu}(\tilde{x}) - \Lambda_k^{[\mu\alpha}\delta^{\nu]\beta}F_{SD}^{\alpha\beta}(x) \simeq F_{SD}^{\mu\nu}(\tilde{x}) - \tfrac{1}{2}k_f^{\mu\nu\alpha\beta}F_{SD}^{\alpha\beta}(x) \quad , \tag{25}$$

that satisfies the modified duality condition (22). Note that the approximation is in fact not necessary in this case, but the notation becomes rather cumbersome for general k_F. The exact solution for the O(3) rotationally invariant component of k_F (which is in fact anti-self-dual) is presented in the appendix.

Next, the case $k_F = \tilde{k}_F$ is considered. This condition implies that k_F has the symmetries of the Weyl conformal tensor with vanishing single traces. In this case, the simple argument given above for anti-self-dual k_F fails because the lower bound on the action in Eq. (21) is not a topological invariant, and is therefore sensitive to small perturbations in the field strengths. In this case, there is no obvious duality condition and the equations of motion must be solved directly for the perturbed instanton solutions. A solution to lowest order in k_F always exists, since the equations reduce to a set of linear, second order elliptic partial differential equations for the gauge fields. The propagators for spin-1 particles in instanton background fields have

7

been previously constructed [13] and are exactly what is needed to formally solve the equations. An explicit example is presented in the next section.

For general k_F, the perturbed field strength may be written as a small perturbation of either the F_{SD} or the F_{ASD} solutions. Remarkably, the approximate value of the action is the same as the conventional case. For example, an instanton that is close to self-dual yields an action of

$$S \simeq \tfrac{1}{2} \int d^4 x Tr(F^2 + k_F^{\mu\nu\alpha\beta} F_{SD}^{\mu\nu} F_{SD}^{\alpha\beta}) \quad . \tag{26}$$

The first term is the conventional action and is invariant to lowest order in any perturbation of the fields due to the fact that the action is at an extremum for the self dual solutions. The O(4) symmetry of the conventional self-dual solutions imply that the second term must vanish, since only observer Lorentz-invariant components of k_F can contribute after the trace is performed. These terms are zero due to the Lorentz-violating nature of k_F. The same arguments apply to the instantons that are close to the anti-self-dual solutions. The numerical value of the action to leading order in k_F is therefore given by the conventional formula

$$S \simeq (8\pi^2/g^2)|q| \quad , \tag{27}$$

for the general case involving arbitrary k_F values. This argument can also be generalized to nonrenormalizable corrections to the pure Yang-Mills sector involving powers of the curvature tensor. This works because any higher order Lorentz-violating corrections must vanish when the O(4) symmetric solutions are substituted into the action. As mentioned previously, any realistic theory of noncommutative gauge fields is argued to be equivalent to a subset of the SME [6], therefore it can be inferred that any realistic theory of noncommutative Yang-Mills fields should not affect the value of the Euclidean action for the instantons to lowest order in the noncommutative, Lorentz-violating $\theta^{\mu\nu}$ parameters.

V. Instantons in SU(2)

To analyze instanton structure, an explicit map is constructed from the asymptotic three sphere S^3 of Euclidean space into the Yang-Mills gauge group G. The

winding number of this map determines the topological charge and therefore the general instanton structure according to the lower bound of the action in Eq (21). For any simple Lie group G, a theorem by Bott [12] proves that any mapping of S^3 into G can be continuously deformed into a mapping into an SU(2) subgroup of G. It is therefore sufficient to fix SU(2) as the gauge group to construct explicit solutions that will exhibit the generic effect of Lorentz violation on the instanton structure.

Here we work with the explicit solutions for $q = 1$, or unit topological charge. The conventional solutions are denoted using the self-dual, antisymmetric tensor $\tau^{\mu\nu}$, where $\tau^{0i} \equiv \sigma^i$ and $\tau^{ij} \equiv \epsilon^{ijk}\sigma^k$, in terms of the conventional Pauli matrices σ^i. This definition provides an explicit embedding of SU(2)→SU(2)×SU(2) which is isomorphic to O(4). The commutation relations

$$[\tau^{\mu\nu}, \tau^{\alpha\beta}] = 2i(\delta^{\mu\alpha}\tau^{\nu\beta} - \delta^{\mu\beta}\tau^{\nu\alpha} - \delta^{\nu\alpha}\tau^{\mu\beta} + \delta^{\nu\beta}\tau^{\mu\alpha}) \quad, \tag{28}$$

and trace relations

$$Tr(\tau^{\mu\nu}\tau^{\alpha\beta}) = 2(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\alpha} + \epsilon^{\mu\nu\alpha\beta}) \quad, \tag{29}$$

follow from the above definition. These quantities may also be expressed using the relation $\tau^{\mu\nu} = i(\tau^\nu \tau^{\mu\dagger} - \delta^{\mu\nu})$, where $\tau^\mu \equiv (i, \vec{\sigma})$. The self-dual gauge field corresponding to $q = 1$ can be expressed as

$$A_{SD}^\mu = -\frac{\tau^{\mu\nu}x^\nu}{g(\rho^2 + x^2)} \quad, \tag{30}$$

and the associated field strength is

$$F_{SD}^{\mu\nu} = \frac{2\rho^2}{g(\rho^2 + x^2)^2}\tau^{\mu\nu} \quad. \tag{31}$$

The parameter ρ determines the instanton size, while the center of the instanton is taken to be at the origin for simplicity. The anti-self-dual solutions are the parity transform of the above fields. These can be expressed using $\tilde{x} = (x^0, -\vec{x})$ as $A_{ASD}^0(x) = A_{SD}^0(\tilde{x})$, $A_{ASD}^i(x) = -A_{SD}^i(\tilde{x})$, $F_{ASD}^{0i}(x) = -F_{SD}^{0i}(\tilde{x})$, and $F_{ASD}^{ij}(x) = F_{SD}^{ij}(\tilde{x})$. This transformation may also be implemented by the transformation $\tau^{\mu\nu} \to \overline{\tau}^{\mu\nu}$ defined by $\tau^{0i} \to \overline{\tau}^{0i} = -\tau^{0i}$, and $\tau^{ij} \to \overline{\tau}^{ij} = \tau^{ij}$. A useful expression for this quantity is $\overline{\tau}^{\mu\nu} = i(\tau^{\nu\dagger}\tau^\mu - \delta^{\mu\nu})$.

For the case $k_F = -\tilde{k}_F$, the modified solutions have already been expressed using Eq.(24) and do not require more explicit computation. For the case $k_F = \tilde{k}_F$, the field equations (14) and (15) must be solved directly since no obvious duality condition can be determined from Eq (21) due to the non-invariant lower bound. To accomplish this, the vector potential is expanded as a perturbation of the self-dual[6] solution $A = A_{SD} + A_k$ and the linear terms in A_k are retained in the equation of motion. The Bianchi identity (15) is automatically satisfied and the equations of motion (14) become (in the Lorentz gauge $\partial^\mu A^\mu = 0$)

$$[D_{SD}^\nu, [D_{SD}^\nu, A_k^\mu]] + 2ig[F_{SD}^{\mu\nu}, A_k^\nu] - ig[D_{SD}^\mu, [A_{SD}^\nu, A_k^\nu]] = j_k^\mu \quad , \tag{32}$$

where

$$j_k^\mu \equiv k_F^{\mu\nu\alpha\beta}[D_{SD}^\nu, F_{SD}^{\alpha\beta}] \quad , \tag{33}$$

and $D_{SD}^\mu \equiv \partial^\mu + ig A_{SD}^\mu$ is the covariant derivative in the conventional self-dual instanton background.

This equation can be solved by performing a convolution of j_k with the corresponding propagator for spin-1 particles in an external instanton field. This propagator has been formally constructed [13], but the explicit form is rather unwieldy and cannot be easily expressed analytically. An alternative approach is adopted here that uses a combination of the propagator approach and a direct substitution technique. First, the solution is studied to lowest order in ρ^2/x^2, corresponding to the asymptotic region far from the self-dual instanton curvature density. This provides the general tensorial structure of the instanton correction that serves as an ansatz for general values of x^2, generating a simple form for the solution to the problem.

It is convenient to perform a gauge transformation to the singular gauge using $U(x) = -ix\cdot\tau^\dagger/x$ so that the potential is better behaved for large x. The transformed potential becomes

$$\overline{A}_{SD}^\mu = -\frac{\rho^2 \overline{\tau}^{\mu\nu} x^\nu}{g x^2 (\rho^2 + x^2)} \quad , \tag{34}$$

---

[6] Only the solution that is close to self-dual is presented here for notational simplicity, the close to anti-self-dual solution may be constructed using an analogous procedure.

with associated field strength

$$\overline{F}_{SD}^{\mu\nu} = \frac{4\rho^2}{g(\rho^2 + x^2)^2} \overline{\tau}^{[\mu\alpha}(\tfrac{1}{4}\delta^{\nu]\alpha} - \frac{x^{\nu]}x^{\alpha}}{x^2}) \quad . \tag{35}$$

In this gauge, the transformed $\overline{j}_k$ is

$$\overline{j}_k^{\mu} = \frac{48\rho^2}{gx^2(\rho^2 + x^2)^3} k_F^{\mu\nu\alpha\beta} \overline{\tau}^{\alpha\gamma} I^{\gamma\nu\beta}(x) \quad , \tag{36}$$

where

$$I^{\gamma\nu\beta} \equiv x^{\gamma}x^{\nu}x^{\beta} - \tfrac{1}{6}x^2(\delta^{\nu\gamma}x^{\beta} + \delta^{\gamma\beta}x^{\nu} + \delta^{\beta\nu}x^{\gamma}) \quad , \tag{37}$$

is a totally symmetric tensor.

The advantage of working in the singular gauge is that the above expressions are all quadratic in ρ. This means that to lowest order in ρ^2/x^2, the propagator may be approximated by the free field Green's function

$$G_0(x, y) = \frac{1}{4\pi^2(x - y)^2} \quad , \tag{38}$$

satisfying $\partial^{\mu}\partial^{\mu}G_0 = -\delta^{(4)}(x - y)$. The perturbed potential to lowest order in ρ^2/x^2 (in the singular gauge) is then given by

$$\overline{A}_k^{\mu} \simeq -\int d^4y \, G_0(x, y)\overline{j}_k^{\mu}(y) \quad . \tag{39}$$

This integral can be performed using standard field theoretic integration techniques. The result of the computation is

$$\overline{A}_k^{\mu} \simeq -\frac{4\rho^2}{gx^6} k_F^{\mu\nu\alpha\beta} \overline{\tau}^{\alpha\gamma} I^{\gamma\nu\beta}(x) \quad . \tag{40}$$

It can be seen that the tensorial structure of $\overline{j}_k$ has been preserved by the convolution with G_0. Some complications arise due to divergent logarithms that cancel out in the computation, but these do not cause theoretical difficulties because the validity of the solution can be verified by direct substitution into the equation of motion. It remains to check that the Lorentz gauge condition is satisfied by this solution. Direct calculation shows that this is the case provided k_F is self-dual, the current case of interest. This indicates that this solution method works for the terms that cannot be removed using a reparametrization of the coordinates.

11

For general values of x^2, an unknown scalar function is included in the expression (40) to produce an ansatz of the form

$$\overline{A}_k^\mu = -\frac{4\rho^2}{g} f(x^2) k_F^{\mu\nu\alpha\beta} \overline{\tau}^{\alpha\gamma} I^{\gamma\nu\beta}(x) \quad . \tag{41}$$

Remarkably, upon substitution into the equation of motion (32), the tensorial structure factors out and the following second order linear differential equation is found for f

$$x^4(\rho^2 + x^2)f'' + 5x^2(\rho^2 + x^2)f' + 3\rho^2 f = -\frac{3}{(\rho^2 + x^2)^2} \quad . \tag{42}$$

This equation has a regular singular point at $x = 0$, causing the homogenoeous solutions to both be badly behaved at the origin. Moreover, any contribution to the homogeneous equation of motion would correspond to a solution to the conventional equations of motion in an instanton background and is therefore not of interest in the present context. On the other hand, the particular solution is well-behaved at the origin as can be verified using the following series expansion for f about $x = 0$

$$f(x^2) = \frac{1}{\rho^6} \sum_{n=0}^{\infty} a_n \left(\frac{x^2}{\rho^2}\right)^n \quad , \tag{43}$$

and expanding the right hand side of Eq. (42) as

$$-\frac{3}{(\rho^2 + x^2)^2} = -\frac{3}{\rho^4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{x^2}{\rho^2}\right)^n \quad , \tag{44}$$

valid for $x^2/\rho^2 < 1$. The resulting recursion relation for the a_n coefficients is

$$a_{n+1} = \frac{3(-1)^n}{n+4} - \frac{n}{n+2} a_n \quad , \tag{45}$$

with $a_0 = -1$. The first few terms gives

$$f(x^2) \approx 1/\rho^6 \left(-1 + \frac{3}{4}\left(\frac{x^2}{\rho^2}\right) - \frac{17}{20}\left(\frac{x^2}{\rho^2}\right)^2 + \frac{37}{40}\left(\frac{x^2}{\rho^2}\right)^3 - \cdots\right) \quad , \tag{46}$$

demonstrating the finite behavior near the origin. For large x^2, a similar expansion in ρ^2/x^2 shows that the function approaches $f(x^2) \to 1/x^6$ as expected.

Transforming the perturbed potential back to the regular gauge yields

$$A_k^\mu \simeq \frac{2\rho^2 x^2}{3g} f(x^2) k_F^{\mu\nu\alpha\beta} \tau^{\alpha\gamma} (\delta^{\gamma\nu} x^\beta + \delta^{\beta\gamma} x^\nu - \delta^{\nu\beta} x^\gamma)] \quad , \tag{47}$$

12

verifying that A_k is zero at the origin in the regular gauge as is required by continuity of the gauge field. The perturbation term behaves asymptotically as $\sim 1/x^3$, and therefore explicitly does not contribute to the the topological charge as expected. The resulting correction to the curvature can be computed, however the specific form is not particularly illuminating. Specifically, there seems to be no obvious generalized duality condition satisfied by F analogous to the situation for $k_F = -\tilde{k}_F$.

VI. SUMMARY

Instantons have long been studied for systems obeying strict Lorentz invariance. In this paper, the structure of Yang-Mills instantons in the presence of small Lorentz-violating background fields that maintain gauge invariance is studied for the first time. No new nonzero action static solutions are present in $n \neq 4$ spatial dimensions as is apparent from Eq (12). The gauge invariance ensures that the conventional pure-gauge asymptotic behavior maintains the same general structure as in the conventional case. This means that conventional arguments can be applied to deduce the quantization of the topological charge. The generality of the SME can then be exploited to infer similar results regarding realistic noncommutative gauge theories.

Specific perturbed instanton solutions for the action considered in this paper are split into two categories that depend on the duality properties of the Lorentz-violating background tensor. For the anti-self-dual k_F case, a reparametrization of the coordinates can be used to construct deformed instantons that satisfy a modified duality condition. The perturbed theory is isomorphic to the conventional Yang-Mills theory in this case so the instanton structure is also isomorphic. The O(3) rotationally invariant term of this class is worked out exactly in the appendix.

When k_F is self-dual, the conventional lower bound argument involving the action fails and the equations of motion must be solved directly. To lowest order in k_F, the resulting equations are linear in the correction to the vector potential and can be formally solved using the Euclidean propagator for a spin-1 particle in an instanton background. For explicit calculation, it turns out to be more practical to first deduce the general tensorial structure in the asymptotic region, then generalize the solution

13

to arbitrary position. General arguments imply that the action is unaltered to lowest order in k_F, but it can be seen from the exact solution given in the appendix that higher order corrections are in general nonzero.

APPENDIX: EXACT SOLUTION FOR O(3) SYMMETRIC CASE

In this appendix, an exact solution (all orders in k_F) for the case of spatial rotationally invariant k_F is presented. In this case, the tensor k_F can be expressed in terms of one independent parameter $\tilde{\kappa}$ as

$$k_F^{0i0j} = -k_F^{i00j} = -k_F^{0ij0} = k_F^{i0j0} = -\frac{\tilde{\kappa}}{2}\delta^{ij} \quad , \tag{A1}$$

and

$$k_F^{ijkl} = \tilde{\kappa}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad . \tag{A2}$$

It is convenient to introduce the notation $\tilde{\kappa} = \sin 2\theta$ and the action takes the form

$$S = \tfrac{1}{2}\int d^4x Tr[F^{\mu\nu}F^{\mu\nu} + \sin 2\theta(F^{ij}F^{ij} - 2F^{0i}F^{0i})] \quad . \tag{A3}$$

To construct the analog of the conventional self-dual solution, consider the following inequality

$$\tfrac{1}{2}\int d^4x Tr\{2[\cos\theta F_-^{0i} - \sin\theta F_+^{0i}]^2 + [\cos\theta F_-^{ij} + \sin\theta F_+^{ij}]^2\} \geq 0 \quad , \tag{A4}$$

with $F_\pm^{\mu\nu} \equiv F^{\mu\nu} \pm \tilde{F}^{\mu\nu}$. This can be rearranged to give the relation

$$S \geq \frac{8\pi^2}{g^2}q\cos 2\theta \quad . \tag{A5}$$

The inequality is saturated when

$$\tilde{F}^{0i} = \frac{1 - \tan\theta}{1 + \tan\theta}F^{0i} \quad , \tag{A6}$$

and

$$\tilde{F}^{ij} = \frac{1 + \tan\theta}{1 - \tan\theta}F^{ij} \quad . \tag{A7}$$

A solution to these equations with $q = 1$ is provided by the gauge potential

$$A^0 = (1 + \tan\theta)A_{SD}^0(\tilde{x}) \quad , \quad A^i = (1 - \tan\theta)A_{SD}^i(\tilde{x}) \tag{A8}$$

where $\tilde{x}^\mu \equiv ((1 + \tan\theta)x^0, (1 - \tan\theta)x^i)$. The resulting field strength is

$$F^{0i} = (1 - \tan^2\theta)F_{SD}^{0i}(\tilde{x}) \quad , \quad F^{ij} = (1 - \tan\theta)^2 F_{SD}^{ij}(\tilde{x}) \quad . \tag{A9}$$

The value of the resulting action can be computed directly from the curvature, yielding the expected value

$$S = \frac{8\pi^2}{g^2} q \cos 2\theta \quad . \tag{A10}$$

In fact, this construction applies to any conventional instanton solution, since the spatially rotational invariant k_F term corresponds to a shift in the speed of light for the gauge fields. It is therefore possible to construct the above solutions by rescaling the time and spatial coordinates appropriately. Note that this does not mean that observable effects are absent, since interactions between the instantons and other particles with conventional Lorentz properties may lead to physical effects. The action is reduced relative to the conventional case by a factor of $\cos\theta$ which is in fact second order in the k_F coefficients. This result is in agreement with general arguments stating that the numerical value of the action is stable to a lowest order perturbation in k_F.

REFERENCES

1. A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. **59B** 85 (1975).

2. M.F. Atiyah, N.J. Hitchin, V. G. Drinfeld, and Yu. I. Manin, Phys. Lett. **65A** 285 (1978).

3. For example, see *Instantons in Gauge Theories*, ed. M. Shifman, World Scientific, Singapore (1994); D. Freed and K. Uhlenbeck, *Instantons and four-manifolds*, New York, Springer-Verlag (1991).

4. For a summary of recent theoretical models and experimental tests see, for example, *CPT and Lorentz Symmetry*, V.A. Kostelecký, ed., World Scientific, Singapore, 1999; *CPT and Lorentz Symmetry II*, V.A. Kostelecký, ed., World Scientific, Singapore, 2002.

5. V.A. Kostelecký and S. Samuel, Phys. Rev. D **39**, 683 (1989); *ibid.* **40**, 1886 (1989); Phys. Rev. Lett. **63**, 224 (1989); *ibid.* **66**, 1811 (1991); V.A. Kostelecký and R. Potting, Nucl. Phys. B **359**, 545 (1991); Phys. Lett. B **381**, 89 (1996); Phys. Rev. D **63**, 046007 (2001); V.A. Kostelecký, M. Perry, and R. Potting, Phys. Rev. Lett. **84**, 4541 (2000).

6. S. Carroll, *et al.*, Phys. Rev. Lett. **87**, 141601 (2001).

7. D. Colladay and V.A. Kostelecký, Phys. Rev. D **55**, 6760 (1997); Phys. Rev. D **58**, 116002 (1998).

8. S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).

9. For a more detailed discussion of concordant frames, see V. A. Kostelecky and R. Lehnert, Phys. Rev. D **63** 065008 (2001).

10. S. Deser, Phys. Lett. **64**, B463 (1976).

11. D. Colladay and P. McDonald, J. Math. Phys. **43** 3554 (2002); A. Kostelecký and M. Mewes, Phys. Rev. D **66**, 056005 (2002).

12. R. Bott, *Bull. Sos. Math. France* **84**, 251 (1956).

13. L. S. Brown, R. Carlitz, D. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1978); H. Levine and L. Yaffe, Phys. Rev. D **19**, 1225 (1979).